Summary of PhD thesis

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In my dissertation I deal with the diadic-harmonic analysis, particularly with the Vilenkin and Vilenkin-like systems. In what follows I will use the usual notations of the Vilenkin analysis.

Let $\mathcal{A}_n$ be the σ-algebra generated by the cosets $I_n(z)$, where $n \in \mathbb{N},\ z \in G_m$. Let $\alpha_j^k, \alpha_n\ (k, j, n \in \mathbb{N})$ be functions satisfying the following conditions:

(i) $\alpha_j^k : G_m \to \mathbb{C}$ is $\mathcal{A}_j$ -measurable $(k, j \in \mathbb{N})$,

(ii) $|\alpha_j^k| := \alpha_0^k := \alpha_j^0 := \alpha_j^k(0) := 1\ (k,\ j \in \mathbb{N})$,

(iii) $\alpha_n := \prod\limits_{j=0}^{\infty} \alpha_j^{n^{(j)}}\ \ (n \in \mathbb{N},\ n^{(j)} := \sum\limits_{k=j}^{\infty} n_k M_k)$.

Let $\chi_n = \psi_n \alpha_n\ (n \in \mathbb{N})$. A function system $\{\chi_n : n \in \mathbb{N}\}$ of this type is called a $\psi\alpha$ *system* on the Vilenkin group G_m.

It is proved that this system is orthonormal and complete in $L^1(G_m)$.

I proved several generalizations with respect to the $\psi\alpha$ and to the double $\psi\alpha$ system.

Let $k \in \mathbb{P}$. We say that $f : \mathbb{N} \to \mathbb{C}$ is an arithmetical function even modulo k, if every $n \in \mathbb{N}$

$$f(n) = f((n, k)),$$

where (n, k) is the greatest common divisor of n and k.

Let

$$\mathcal{B}_k := \{f \text{ even modulo } k\}, \qquad \mathcal{B} := \bigcup_{k \in \mathbf{N}} \mathcal{B}_k.$$

If $f : \mathbf{N} \to \mathbf{R}$ then

$$\overline{M}(f) := \limsup_{x \to \infty} x^{-1} \sum_{1 \le j \le x} f(j)$$

are called the upper mean values of f.

Let $1 \le q < \infty$. Then the upper mean value gives rise to a seminorm

$$\|f\|_q := \{\overline{M}(|f|^q)\}^{1/q}$$

on the linear space $\{f | f : \mathbf{N} \to \mathbf{C}, \|f\|^q < \infty\}$. We denote by $\mathcal{B}^q$ the closure of $\mathcal{B}$ in this norm. Functions in $\mathcal{B}^q$ are called almost even arithmetical functions.

If $n \in \mathbb{N}, n = \sum\limits_{j=0}^{\infty} n_j M_j$, then let

$$\breve{n} := \sum_{j=0}^{\infty} n_j M_{j+1}^{-1}$$

and of course only a finite number of n_j's is not zero.

Denote by

$$\kappa_n(x) := \lim_{k\to\infty} \exp(2\pi\imath\breve{n} \sum_{j=0}^{k} x_j M_j) \ (n \in \mathbb{N}, x \in G_m).$$

There exists the limit, and the system $\{\kappa_n | n \in \mathbb{N}\}$ on G_m is a $\psi\alpha$ system.

If $f \in \mathcal{B}^q$ and $1 \leq q < \infty$, then there exists $f^* \in L^q(G_m)$ μ almost everywhere unique, that $f^*(\lambda^{-1}(\breve{n})) = f(n)$, $\|f^*\|_q = \|f\|_{\mathcal{B}^q}$ and $f_n \to f \Leftrightarrow f_n^* \to f^*$, where λ is the so-called Fine-map.

If for every $k \in \mathbb{P}$ there exists $n = n(k)$ such that $k|\ M_n$, we say that G_m is a R-Vilenkin group.

Theorem *If G_m is a R-Vilenkin group, then there exists F function which is an element of $\mathcal{B}^q$ where $1 \leq q$, the Vilenkin—Fourier series of which diverges everywhere.*

My main results are the following (both for bounded and double systems):

Theorem *The operator σ^* is of type (H°, L) i.e.*

$$\|\sigma^* f\|_1 \leq c\|f\|_{H^\circ} \quad (f \in H).$$

Theorem *Let $f \in L^1(G_m \times G_{\tilde{m}})$. Then $\sigma_{(n_1,n_2)} f \to f$ a.e. as $n_1, n_2 \to \infty$, where $n_1, n_2 \in \mathbb{P}$ and $\beta^{-1} \leq n_1/n_2 \leq \beta$ for some fixed parameter $\beta \geq 1$.*

The other system (also introduced by György Gát) is a common generalization of several well-known systems.

Let us introduce this one on G_m, which will be called Vilenkin-like system. Now G_m is not the Vilenkin group, but the so-called Vilenkin space without any operator.

The complex valued functions $r_k^n : G_m \to \mathbb{C}$ $(k, n \in \mathbb{N})$ we call generalized Rademacher functions, if they have the following properties:

(i.) r_k^n is $\mathcal{A}_{k+1}$ measurable (i.e. $r_k^n(x)$ depends only on $x_0, ..., x_k$ $(x \in G_m)$), $r_k^0 = 1$ for all $k, n \in \mathbb{N}$.

(ii.) If M_k is a divisor of n, l and $n^{(k+1)} = l^{(k+1)}$ $(k, l, n \in \mathbb{N})$, then

$$E_k(r_k^n \bar{r}_k^l) = \begin{cases} 1 \text{ if } n_k = l_k, \\ 0 \text{ if } n_k \neq l_k \end{cases}$$

($\bar{z}$ is the complex conjugate of z).

(iii.) If M_k is a divisor of n (that is, $n = n_k M_k + n_{k+1} M_{k+1} + ... + n_{|n|} M_{|n|}$). Then

$$\sum_{n_k=0}^{m_k-1} |r_k^n(x)|^2 = m_k$$

for all $x \in G_m$

Now define the Vilenkin-like system $\psi := (\psi_n : n \in \mathbb{N})$ as follows

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}}, \quad n \in \mathbb{N}.$$

(Since $r_k^0 = 1$, then $\psi_n := \prod_{k=0}^{|n|} r_k^{n^{(k)}}$). The Vilenkin-like system ψ is orthonormal.

I proved a norm inequality with respect to this system.

Theorem *There exists a $c > 0$ absolute constant that if $f \in H(G_m)$ and the sequence m is bounded, then*

$$\sup_{N \in \mathbb{P}} \frac{1}{\log N} \sum_{n=1}^{N} \frac{\|S_n f\|_1}{n} \leq c\|f\|_H.$$

It is managed to verify the two-dimensional version of this inequality.

Theorem *Assume that $\beta \geq 1$, $f \in H^1(G_m \times G_m)$ and $2 \leq N, K \in \mathbb{P}$. Then there exists a constant $c > 0$ depending only on β such that*

$$\sup_{N,K} \frac{1}{\log N \log K} \sum_{\beta^{-1} \leq n/k \leq \beta (n,k) \leq (N,K)} \frac{\|S_{n,k} f\|_1}{nk} \leq c\|f\|_{H^1}.$$