Summation methods of Walsh-Fourier series, almost everywhere convergence and divergence

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The n-th Walsh-Paley function is $\omega_n(x) := (-1)^{\sum_{i=0}^{\infty} n_i x_i}$, where $n = \sum_{i=0}^{\infty} n_i 2^i \in \mathbb{N}$, $x = \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}} \in [0,1)$ and $n_i, x_i \in \{0,1\}$, $i = 0, 1, \ldots$

The Dirichlet and Fejér kernel functions are defined as

$$D_n := \sum_{k=0}^{n-1} \omega_k, \quad K_n := \frac{1}{n} \sum_{k=0}^{n-1} D_k.$$

Fourier coefficients, partial sums of Fourier series, Fejér means:

$$\hat{f}(n) := \int_0^1 f(x)\omega_n(x)dx,$$

$$S_n f(y) := \sum_{k=0}^{n-1} \hat{f}(k)\omega_k(y) = \int_0^1 f(x \dotplus y) D_n(x)dx,$$

$$\sigma_n f(y) := \frac{1}{n} \sum_{k=0}^{n-1} S_k f(y) = \int_0^1 f(x \dotplus y) K_n(x)dx.$$

Fejér means

In 1926 Lebesgue proved for the trigonometric system, $\sigma_n f(x) \to f(x)$ for a.e. x for all integrable function.

It is proved for the Walsh-Paley system by Fine [2] and for the Walsh-Kaczmarz system it is due to Gát [4]. The Walsh-Kaczmarz system is defined as follows:

Let $\kappa_0 = 1$ and for $n \geq 1$, $|n| = \lfloor \log_2 n \rfloor$ define

$$\kappa_n(x) := r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

That is,

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{\omega_k : 2^k \leq n < 2^{k+1}\}$$

for all $k \in \mathbb{N}$. In other words, the Walsh-Paley and the Walsh-Kaczmarz systems are dyadic blockwise rearrangement of each other.

The (C, α) Cesàro means of the integrable function f are

$$\sigma_n^{\alpha} f(y) := \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f(y),$$

where $A_k^{\alpha} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}{k!} \quad (\alpha \neq -k)$.

Fine proved [2] for the Walsh-Paley system and each $\alpha > 0$, $f \in L^1$ the a.e. convergence $\sigma_n^{\alpha} f \to f$.

The Walsh-Kaczmarz analogue is due to Simon, [13].

For Walsh-Paley Fejér kernel functions we have $K_n(x) \to 0$ $(n \to \infty)$ for every $x \neq 0$. However, they can take negative values which is a different situation from the trigonometric case.

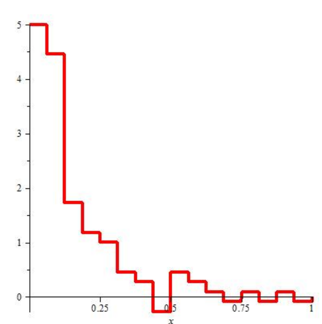

Figure 1: K_{11} *Walsh-Paley-Fejér kernel*

On the other hand, for Walsh-Kaczmarz Fejér kernel functions we have $K_n(x) \to \infty$ $(n \to \infty)$ at every dyadic rational.

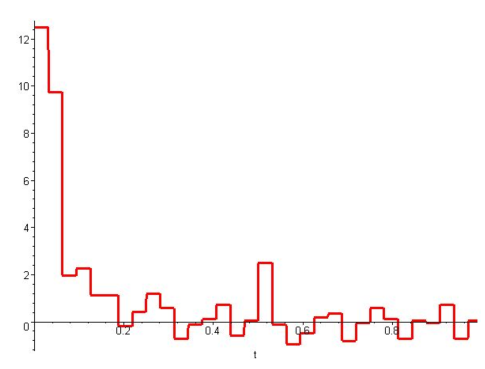

Figure 2: K_{26} *Walsh-Kaczmarz-Fejér kernel*

Summation of subsequence of partial sums

In 1936 Zalcwasser [17] asked how "rare" the sequence of strictly monotone increasing integers $a(n)$ can be such that

$$\frac{1}{N} \sum_{n=1}^{N} S_{a(n)} f \to f \qquad (1)$$

"in some sense". For trigonometric system it is solved for continuous functions (uniform convergence) by Salem [12]: If the sequence a is convex, then the condition $\sup_n n^{-1/2} \log a(n) < +\infty$ is necessary and sufficient for the uniform convergence for every continuous function. For the time being, this issue with respect to the Walsh-Paley system has not been solved. Only, a sufficient condition is known, which is the same as in the trigonometric case (Glukhov [8]).

With respect to convergence almost everywhere and integrable functions the situation is more complicated. Belinsky proved [1] in the case of the trigonometric system the existence of a sequence $a(n) \sim \exp(\sqrt[3]{n})$ such that the relation (1) holds a.e. for every integrable function. In this paper Belinsky also conjectured that if the sequence a is convex, then the condition $\sup_n n^{-1/2} \log a(n) < +\infty$ is necessary and sufficient again. So, that would be the answer for the problem of Zalcwasser in this point of view (trigonometric system, a.e. convergence and L^1 functions). We proved in the case for the Walsh-Paley system the following two results.

Theorem. (Gát, [6]) Let $a : \mathbb{N} \to \mathbb{N}$ be a sequence with property $\frac{a(n+1)}{a(n)} \geq q > 1$ $(n \in \mathbb{N})$. Then for all $f \in L^1([0,1))$ we have the a.e. relation

$$\frac{1}{N} \sum_{n=1}^{N} S_{a(n)} f \to f.$$

Theorem. (Gát, [6]) Let $a : \mathbb{N} \to \mathbb{N}$ be a convex sequence with property $a(+\infty) = +\infty$. Then for each $f \in L^1([0,1))$ we have the a.e. relation

$$\frac{1}{\log N} \sum_{n=1}^{N} \frac{S_{a(n)} f}{n} \to f.$$

Two-dimensional functions

Two-dimensional Fourier coefficients of two variable integrable function $f : [0,1) \times [0,1) \to \mathbb{C}$ are

$$\hat{f}(k,n) := \int_0^1 \int_0^1 f(x,y)\omega_k(x)\omega_n(y)dxdy.$$

The rectangular partial sums of the two-dimensional Fourier series

$$S_{M,N} f(x,y) := \sum_{k=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(k,n)\omega_{k,n}(x,y).$$

The two-dimensional Fejér means

$$\sigma_{M,N} f(x,y) := \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{n=0}^{N-1} S_{k,n} f(x,y).$$

How can a function be reconstructed, knowing only its Walsh-Fourier coefficients? Under what conditions (and in what sense) $\sigma_{M,N} f(x,y) \to f(x,y)$? In the trigonometric case we mention two historical results.

- In 1935 Jessen, Marcinkiewicz and Zygmund: $\sigma_{M,N} f \to f$ a.e., as if $\min\{m,n\} \to \infty$ for $f \in L^1 \log^+ L^1$,
- In 1939 Marcinkiewicz and Zygmund: $\sigma_{M,N} f \to f$ a.e., as if $1/\beta \leq M/N \leq \beta$ ($\beta \geq 1$ fixed).

For the two-dimensional Fejér means in the Walsh-Paley and Walsh-Kaczmarz case it is known:

- Móricz, Schipp, and Wade [10]: The unconditional (Pringsheim sense) convergence for functions in $L \log^+ L$, and the restricted one for L^1 functions $\sigma_{2^N, 2^M} f \to f$ with $|N - M| \leq C$.
- Weisz [15] and Gát [3] the L^1 situation for all index pairs, not only powers of two.
- Simon [14] for the Walsh-Kaczmarz system, both situation (Pringsheim and restricted convergence).

The convergence space $L \log^+ L$ is maximal in the following sense: Gát [5] proved the following divergence result:

Let $\delta : [0, +\infty) \to [0, +\infty)$ be measurable, and $\lim_{t \to \infty} \delta(t) = 0$. Then $\exists f \in L^1([0,1)^2)$ such as $f \in L\log^+ L\delta(L)$, and $\sigma_{n_1,n_2} f$ does not converge to f a.e. as $\min(n_2, n_2) \to \infty$.

Marcinkiewicz means

Marcinkiewicz means for $f \in L^1([0,1)^2)$:

$$t_n f(x) := \frac{1}{n} \sum_{k=0}^{n-1} S_{k,k} f(x).$$

Marcinkiewicz [9] proved for functions belonging to $L \log^+ L$ a.e. relation $t_n f \to f$ with respect to the trigonometric system. The "L^1 result" for the trigonometric, Walsh-Paley, Walsh-Kaczmarz systems:

- Zhizhiasvili [18] (trigonometric system),
- Weisz [16], (Walsh system),
- Nagy [11] (Walsh-Kaczmarz system).

Define the Marcinkiewicz-like means [7]: ($|n| = \lfloor \log_2 n \rfloor$), $\alpha = (\alpha_1, \alpha_2) : \mathbb{N}^2 \to \mathbb{N}^2$

$$t_n^{\alpha} f(x) := \frac{1}{n} \sum_{k=0}^{n-1} S_{\alpha_1(|n|,k), \alpha_2(|n|,k)} f(x).$$

The following two conditions on α play a prominent role.

$$\#\left\{l \in \mathbb{N} : \alpha_j(|n|, l) = \alpha_j(|n|, k), l < n\right\} \leq C \, (k < n),$$

$$\max\left\{\alpha_j(|n|, k) : k < n\right\} \leq Cn \quad (n \in \mathbb{P}, j = 1, 2).$$

For the Walsh-Paley system we have:

Theorem of convergence , Gát, [7]: Let α satisfy the two conditions above. Then $t_n^{\alpha} f \to f$ a.e. $(f \in L^1)$.

Theorem of divergence, Gát, [7]: Let $\gamma : \mathbb{N} \to \mathbb{N}$ be any function with property $\gamma(+\infty) = +\infty$. Then there exists a function $f \in L^1([0,1)^2)$ and α satisfying first condition and $\max\{\alpha_1(|n|, k) : k < n\} \leq Cn$, $\max\{\alpha_2(|n|, k) : k < n\} \leq Cn\gamma(n)$ such that $\limsup_{n \in \mathbb{N}} |t_n^{\alpha} f| = +\infty$ almost everywhere.

Corollary (Gát [7])

$$\frac{1}{2^n} \sum_{k=0}^{2^n-1} S_{\alpha_1(n,k), \alpha_2(n,k)} f(x) \to f$$

a.e. for all $f \in L^1$, where $\alpha_j(n,k) \leq C2^n$. This issue is open for Walsh-Kaczmarz, trigonometric systems.

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