Restricted two-dimensional Walsh-Fejér means and generalizations

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1. Earlier results on trigonometric and Walsh system

In 1939 for double trigonometric Fourier series Marcinkiewicz and Zygmund proved the convergence almost everywhere of Fejér means $\sigma_n f$ of integrable functions, where the set of indices is inside a positive cone around the identical function, that is

$$\beta^{-1} \le n_1/n_2 \le \beta$$

is provided with some fixed parameter $\beta \ge 1$ [10]. Moreover, Jessen, Marcinkiewicz, Zygmund proved the convergence $\sigma_n f \to f$ almost everywhere without any restriction on the indices, but only for functions in $L \log^+ L$ [9]. In 2007, Gát gave a common generalization of these two result [2]. He defined the concept of cone-like sets by the help of cone restricted functions (CRF). For more details see Theorem 1, 2, 5 and Corollary 6.

Weisz extended this result to higher dimensions, to Cesàro and Riesz means and proved that the maximal operator is bounded from the Hardy space H_p to the space L_p for $p > p_0 = \max\{1/(\alpha_j + 1) : j = 1, ..., d\}$ [18].

In 1992, for double Walsh-Fourier series Móricz, Schipp and Wade proved that $\sigma_n f$ converge to f a.e. in the Pringsheim sense (that is, no restriction on the indices other than $\min(n_1, n_2) \to \infty$) for all functions $f \in L \log^+ L$ [11]. Later, Gát proved that the theorem of Móricz, Schipp and Wade can not be sharpened. Namely, let $\delta : [0, +\infty) \to [0, +\infty)$ be a measurable function with property $\lim_{+\infty} \delta = 0$, then there exists a function $f \in L \log^+ L\delta(L)$ such that $\sigma_n f$ does not converge to f a.e. as $\min(n_1, n_2) \to \infty$ [3].

In 1996, Gát, Weisz proved the convergence almost everywhere for Fejér means $\sigma_n f$ of integrable functions, where the set of indices is inside a positive cone [4, 19].

2. Representative product systems

Let $m := (m_k, k \in \mathbb{N})$ be a sequence of positive integers such that $m_k \ge 2$ and G_k be a finite group with order m_k, $(k \in \mathbb{N})$ (it is important to note that G_k could be a non-commutative group, as well). Suppose that each group has discrete topology and normalized Haar measure μ_k.

Let G_m be the compact group formed by the complete direct product of the groups G_k with the product of the topologies, operations and measures (μ). G is called a bounded group if the sequence $m = (m_k, k \in \mathrm{N})$ is bounded.

Let ψ be the *representative product system* on G_m. The system ψ is the character system of G_m. For more details see [8].

We have to note that the representative product system contains the Walsh and Vilenkin system as special case.

3. Vilenkin spaces

Let $m = (m_k : k \in \mathbb{N})$ be a sequence of natural numbers, which members are greater than 1. Let G_{m_k} be a set of cardinality m_k. We supply each set G_{m_k} with the discrete topology and the measure μ_k which is defined by $\mu_k(\{i\}) = 1/m_k$ for all $i \in G_{m_k}$. Let G_m be the compact set formed by the complete direct product of the sets G_{m_k} with the product measure and the product topology.

G_m is called a Vilenkin space. It is important to note that *there is no operation defined on the Vilenkin space.*

The *Vilenkin-like system* $\psi := (\psi_n : n \in \mathbb{N})$ by

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}}$$

where r_k^n is the generalized Rademacher system. For the original definition see [5].

The Vilenkin-like system is a generalization of the Walsh and Vilenkin system, the character system of the group of 2-adic integers, the representative product systems, the UDMD (unitary dyadic martingale difference) systems and the UCP (universal contractive projection) system introduced by Schipp and other systems.

4. Cone-like sets

Let $\alpha : [1, +\infty) \to [1, +\infty)$ be a strictly monotone increasing continuous function with property $\lim_{+\infty} \alpha = +\infty$, $\alpha(1) = 1$, and $\beta : [1, +\infty) \to [1, +\infty)$ be a monotone increasing function with property $\beta(1) > 1$.

Let us define the *cone-like restriction sets*:

$$\mathbb{N}_{\alpha,\beta,1} := \left\{ n \in \mathbb{N}^2 : \frac{\alpha(n_1)}{\beta(n_1)} \le n_2 \le \alpha(n_1)\beta(n_1) \right\},$$

$$\mathbb{N}_{\alpha,\beta,2} := \left\{ n \in \mathbb{N}^2 : \frac{\alpha^{-1}(n_2)}{\beta(n_2)} \le n_1 \le \alpha^{-1}(n_2)\beta(n_2) \right\}.$$

Let $\beta(x) = \beta$ be a constant function.

For $i = 1, 2$ set

$$\mathbb{N}_{\alpha,i} := \{\mathbb{N}_{\alpha,\beta,i} : \beta > 1\}.$$

For a fixed $i \in \{1, 2\}$, $\mathbb{N}_{\alpha,i}$ is said to be weaker than $\mathbb{N}_{\alpha,3-i}$, if for all $L \in \mathbb{N}_{\alpha,i}$, there exists an $\tilde{L} \in \mathbb{N}_{\alpha,3-i}$ such that $L \subset \tilde{L}$. (It is denoted by $\mathbb{N}_{\alpha,i} \prec \mathbb{N}_{\alpha,3-i}$)

If $\mathbb{N}_{\alpha,1} \prec \mathbb{N}_{\alpha,2}$ and $\mathbb{N}_{\alpha,2} \prec \mathbb{N}_{\alpha,1}$, then we say that $\mathbb{N}_{\alpha,1}$ and $\mathbb{N}_{\alpha,2}$ are equivalent and denote this by $\mathbb{N}_{\alpha,1} \sim \mathbb{N}_{\alpha,2}$.

The function α is called a *cone-like restriction function* (CRF), if $\mathbb{N}_{\alpha,1} \sim \mathbb{N}_{\alpha,2}$.

Let us set $\mathbb{N}_\alpha := \mathbb{N}_{\alpha,1} \cup \mathbb{N}_{\alpha,2}$.

A function α is CRF if and only if there exist $\zeta, \gamma_1, \gamma_2 > 1$ such that

$$\gamma_1 \alpha(x) \le \alpha(\zeta x) \le \gamma_2 \alpha(x)$$

holds for each $x \ge 1$.

Let us define the maximal operator

$$\sigma_L^* f := \sup_{n \in L} |\sigma_n f|.$$

The next theorem was proven for trigonometric system by Gát [2], for Walsh-Paley system by Gát and Nagy [6], for representative product systems [12] and Vilenkin-like systems [13] by Nagy. The main tool is a special type of Calderon-Zygmund decomposition lemma.

Theorem 1 *Let α be CRF, $L \in \mathbb{N}_\alpha$. Then the operator σ_L^* is of weak type $(1,1)$.*

Theorem 2 *Let α be CRF, $L \in \mathbb{N}_\alpha$. Then for any $f \in L^1$ the equality*

$$\lim_{\substack{\wedge n \to \infty \\ n \in L}} \sigma_n f = f$$

holds a.e.

If we choose $\alpha(x) := x$ and $\beta(x) = \beta$ (where $\beta \ge 1$ is a constant) then we get the next corollary of Theorem 2. It was proven for trigonometric system by Marcinkiewicz and Zygmund [10], for Walsh system by Weisz [19] and Gát [4], for Vilenkin system (ψ_α system) Gát and Blahota [1], for Walsh-Kaczmarz system by Simon [16].

Corollary 3 *Let $f \in L^1$ and $\beta \ge 1$ be a fixed parameter. Then the relation*

$$\lim_{\substack{\wedge n \to \infty \\ \beta^{-1} \le n_1/n_2 \le \beta}} \sigma_n f = f \quad \text{a.e.}$$

holds.

Corollary 4 *Let α be CRF and $L \in \mathbb{N}_\alpha$. Then the operator σ_L^* is of type (p,p) for all $1 < p \le \infty$.*

The next theorem was proven for trigonometric system by Gát [2], for Walsh-Paley system by Gát and Nagy [6].

Theorem 5 *Let α be CRF, $\beta : [1, +\infty) \to [1, +\infty)$ be a monotone increasing function with property $\lim_{+\infty} \beta = +\infty$, and $\delta : [1, +\infty) \to [0, +\infty)$ be a measurable function with property $\lim_{+\infty} \delta = 0$. Let $L := \mathbb{N}_{\alpha,\beta,1}$ or $L := \mathbb{N}_{\alpha,\beta,2}$. Then there exists a function $f \in L^1 \log^+ L\delta(L)$ such that*

$$\limsup_{\substack{\wedge n \to \infty \\ n \in L}} \sigma_n f = +\infty \quad \text{holds a.e.}$$

Corollary 6 *Let α be CRF, $\beta : [1, +\infty) \to [1, +\infty)$ be a monotone increasing function with property $\beta(1) > 1$, and $L := \mathbb{N}_{\alpha,\beta,1}$ or $L := \mathbb{N}_{\alpha,\beta,2}$. Then*

$$\limsup_{\substack{\wedge n \to \infty \\ n \in L}} \sigma_n f = +\infty$$

holds a.e. for all $f \in L^1$ if and only if the function β is not bounded.

5. Hardy spaces

In 2011, Weisz defined a new type martingale Hardy space depending on the function α [17]. The original definition of Weisz is given for dimension d and for Vilenkin group. Here we present the two-dimensional version of it and for Walsh group only .

For a given $n_1 \in \mathbb{N}$ set $n_2 := \lfloor \log_2 \alpha(2^{n_1}) \rfloor$, that is, n_2 is the order of $\alpha(2^{n_1})$ (this means that $2^{n_2} \le \alpha(2^{n_1}) < 2^{n_2+1}$). Let $\overline{n}_1 := (n_1, n_2)$.

We have a class of one-parameter martingales $f = (f_{\overline{n}_1},\ n_1 \in \mathbb{N})$ with respect to the σ-algebras $(\mathcal{F}_{\overline{n}_1},\ n_1 \in \mathbb{N})$. The σ-algebra is generated by the 2-dimensional rectangles $I_{n_1}(x^1) \times I_{n_2}(x^2)$ ($(x^1, x^2) \in G^2$).

The maximal function of a martingale f is defined by

$$f^* = \sup_{n_1 \in \mathbf{N}} \left| f_{\overline{n}_1} \right|.$$

For $0 < p < \infty$ the *martingale Hardy space $H_p^\alpha(G^2)$* consists of all martingales for which $\|f\|_{H_p^\alpha} := \|f^*\|_p < \infty$.

Here we give the next two theorems for two-dimensional Walsh-Fejér means only.

The next theorem belongs to Weisz for Walsh system [17].

Theorem 7 *The maximal operator σ_L^* is bounded from H_p to L_p for $1/2 < p < \infty$ and is of weak type $(1,1)$.*

Recently, the next theorem is reached by the author [14].

Theorem 8 *Let α be CRF. The maximal operator σ_L^* is not bounded from the Hardy space $H_{1/2}^\alpha$ to the space $L_{1/2}$.*

We mention that Weisz reached his result for (C, α) means of d-dimensional Vilenkin-Fourier series.

Recently, Theorem 7 and 8 are proven for Walsh-Kaczmarz system by the author [15].

If we choose $\alpha(x) = x$ the identical function and $\beta(x) = \beta$ a constant function (where $\beta \ge 1$), then Theorem 7 is reached for Walsh-Paley system by Weisz [19], for Walsh-Kaczmarz system by Simon [16] and for the character system of 2-adic integers by Gát and Nagy [7].

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