Summary of PhD thesis

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The theory of abstract harmonic analysis has been a relevant progress in the last decades. An increasing number of mathematicians have adopted the point of view that the most appropriate setting for the development of the theory of Fourier analysis is furnished by the class of all locally compact groups. Starting of the classical theory of Fourier series and integrals the relative ease with which the basic concepts and theorems can be transferred to this general context in the abelian case is not valid for the non-commutative case. For instance, it is well known that the Riemann-Lebesgue lemma is not valid for non-commutative cases.

The structure of topological groups was extensively studied in the years 1925-1940, and the subject is far from dead even today. The study of the direct products of topological groups have been started since the beginning of the theory of topological groups. Pontryagin examined very extensively the structure of countable direct products treated special cases of finite direct products. Vilenkin obtained several results for the commutative cases.

The dyadic group is the simplest but nontrivial model of the complete product of finite groups. Representing the characters of the dyadic group ordered in the Paley's sense, we obtain the Walsh system. This system is applied in the processing of data but has an interesting theoretical point of view.

A natural generalization on the Walsh dyadic system is the Vilenkin system introduced by Vilenkin in 1947. He used the set of all characters of the complete product of arbitrary cyclic groups to obtain the commutative case. In Hungary a dyadic analysis team works leaded by Schipp having several results in this theory. For instance, they proved that the Paley theorem is true for an arbitrary Vilenkin group, i.e. the partial sums of the Vilenkin-Fourier series of a function in $L^p(G)$ $(1 < p < \infty)$ converge in the appropriate norm to the function (Young, Schipp, Simon).

The example above is not true for all cases if we take the complete product of arbitrary finite group (not necessarily commutative). The study of this groups is the aim of this work. These studies were appeared in the article by Gát and Toledo first and they obtained not only negative results for this groups, because they also proved the convergence in L^p-norm of the Fejér means of Fourier series when $p \geq 1$ in the bounded case.

This work is organized as follows. The first chapter is introductory, introducing the topology, the measure and the system with which we work. This kind of system is called (by the author) a representative product system because we use representation theory to collect the functions appeared in it. The Weyl-Peter's theorem ensure the orthonormality and completeness in L^2 of this system. Representing this system on the interval $[0, 1]$ we plot and show relevant examples.

We introduce the basic concepts of Fourier analysis and give the properties of the Dirichlet kernels to study the convergence in norm of Fourier series and Fejér means. We also introduce the concept of modulus of continuity to give class of functions for which the partial sums of it's Fourier series converge to the function in L^1 or in the uniform norm. Finally, we obtain an important positive result, i.e. if G is a bounded group, the Fejér means of a function $f \in L^p(G)$, $1 \leq p \leq \infty$ converge to the function in L^p-norm.

We estimated the Fourier coefficients which not necessarily tend to zero, using the modulus of continuity of the function and the uniform norm of the system. We specially study the functions with bounded fluctuation. On the other hand we consider an interesting class of functions, namely the ones that are constant on every conjugacy classes. The system of characters of the representations is complete in this class of functions, so we use characters to study the absolute convergence of series constructed in this way.

We treats the general case of product system adapting the results of F. Schipp for the convergence in Hardy and BMO norms. We use the convergence of operators with property Δ to study the conjugate martingale transforms defined on not necessarily bounded Vilenkin group. In this chapter the notation of the different Hardy and BMO spaces is the same to the notation appeared in the book written by F. Weiss.