Some properties of kernels with respect to Vilenkin-like systems

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1. Definitions, notation

The concept of the Vilenkin-like space was introduced by G. Gát in 2001 [5]. Let $m := (m_0, m_1, ...)$, where $1 < m_k \in \mathbb{P}$. Denote by G_{m_k} a set, where the number of the elements is m_k $(k \in \mathbb{N})$ with the discrete topology. Define the measure on G_{m_k} as

$$\mu_k(\{j\}) := \frac{1}{m_k} \quad (j \in G_{m_k}, k \in \mathbb{N}).$$

Let $G_m := \mathop{\times}_{j=1}^{\infty} G_{m_j}$, with the product of the topologies and measures (denoted by μ). This product measure is a regular Borel one on G_m with $\mu(G_m) = 1$. The name of G_m is the *Vilenkin-like space*. We can talk about bounded or unbounded Vilenkin space, according as the sequence m is bounded or not. The elements of G_m are $x := (x_0, x_1, \dots)$ $(x_k \in G_{m_k})$.

A neighbourhood base of G_m (for $x \in G_m,\ n \in \mathbb{P}$) is given by

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m | y_0 = x_0, ..., y_{n-1} = x_{n-1}\}.$$

The concept of *generalized number system* is well-known. Let $M_0 := 1, M_{k+1} := m_k M_k$ $(k \in \mathbb{N})$, then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, where $n_k \in \{0, 1, \dots, m_k - 1\}$ $(k \in \mathbb{N})$, and only a finite number of n_k's differ from zero.

Let $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ if $n \in \mathbb{P}$, and $|0| := 0$. Let $n^{(k)} = \sum_{j=k}^{\infty} n_k M_k$.

Denote by $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)\,(x \in G_m,\ n \in \mathbb{N})$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n\,(n \in \mathbb{N})$.

The complex valued functions $r_k^n : G_m \to \mathbb{C}$ $(k, n \in \mathbb{N})$ are called *generalized Rademacher functions*, if they have these four properties:

i. $r_k^n(x)$ $(k, n \in \mathbb{N})$ is $\mathcal{A}_{k+1}$ measurable and $r_k^0 = 1$.

ii. If M_k is a divisor of n, l and $n^{(k+1)} = l^{(k+1)}$ $(k, l, n \in \mathbb{N})$, then

$$E_k(r_k^n \bar{r}_k^l) = \begin{cases} 1 \text{ if } n_k = l_k, \\ 0 \text{ if } n_k \neq l_k. \end{cases}$$

iii. If M_k is a divisor of n, then for all $x \in G_m$

$$\sum_{n_k=0}^{m_k-1} |r_k^n(x)|^2 = m_k.$$

iv. There exists a $\delta > 1$, for which $\|r_k^n\|_\infty \leq \sqrt{m_k/\delta}$ for all $k, n \in \mathbb{N}$.

Now define the *Vilenkin-like system* ψ as follows $\psi := (\psi_n : n \in \mathbb{N})$, where

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}} \quad (n \in \mathbb{N}).$$

Define the Dirichlet and the Fejér kernels in the usual way

$$D_n(y, x) := \sum_{k=0}^{n-1} \psi_k(y)\overline{\psi}_k(x) \qquad (n \in \mathbb{P},\ D_0(y, x) := 0),$$

$$K_n(y, x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(y, x) \qquad (n \in \mathbb{P}).$$

In some cases of systems above (denoted by ϑ now) Dirichlet and Fejér kernel functions depend only on one element of the domain. The "one way" connection between the two concepts is $D_n(y, x) = D_n^{\vartheta}(y - x)$.

Finally, let us introduce the maximal value sequences of the Dirichlet and Fejér kernels in the following way

$$D_n := \sup_{x,y \in G_m} |D_n(y, x)| \qquad (n \in \mathbb{N}),$$

$$K_n := \sup_{x,y \in G_m} |K_n(y, x)| \qquad (n \in \mathbb{P}).$$

2. Examples

Let us see some known examples for the Vilenkin-like system.

1. The Walsh-Paley and Vilenkin systems [8],[1].
2. The group of 2-adic (m-adic) integers and the group of characters [7], [9].
3. The product system of coordinate functions of unitary irreducible representations of non-commutative Vilenkin groups (in this case the group G_m is the Cartesian product of any finite groups) [10].
4. A system in the field of number theory. The so-called $\psi\alpha$ Vilenkin-like system (on Vilenkin groups) was a new tool in order to investigate limit periodic arithmetical functions [4].
5. The UDMD product system [8].

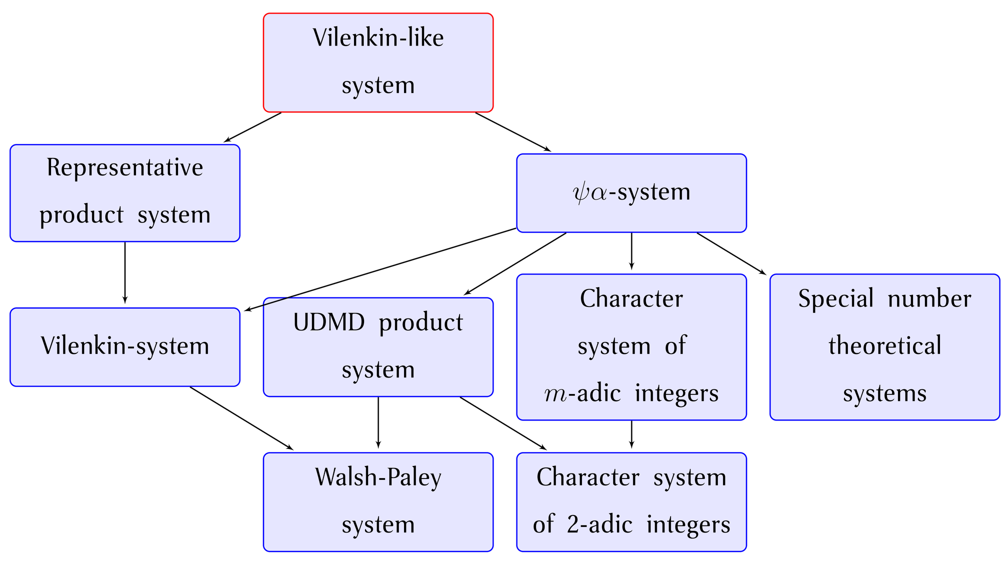

Table 1. Systems and their connections

3. Results

The Vilenkin-like system ψ is orthonormal and complete on G_m [5]. It is known for bounded Vilenkin-like systems [5], that if $f \in L^1(G_m)$, then $S_{M_n} f \to f$ a. e, $\sigma_n f \to f$ a. e. and if $f \in L^p(G_m), 1 \leq p < \infty$, then $\sigma_n f \to f$ in L^p-norm.

We could prove, that for kernels the supremums are actually in the diagonal.

Theorem 1 *[2]*

$$D_n = \sup_{x \in G_m} D_n(x, x) \qquad (n \in \mathbb{N}),$$

$$K_n = \sup_{x \in G_m} K_n(x, x) \qquad (n \in \mathbb{P}).$$

The Paley lemma for Vilenkin-like system is true [5]. Namely, if $x, y \in G_m,\ n \in \mathbb{N}$, then

$$D_{M_n}(y, x) = \begin{cases} M_n \text{ if } y \in I_n(x), \\ 0 \quad \text{if } y \notin I_n(x). \end{cases}$$

From this result $D_{M_n} = M_n$ $(n \in \mathbb{N})$ [2] is trivial.

We can estimate maximal value sequences of kernels in the following way.

Theorem 2 *[2]*

$$n \leq D_n \leq M_{|n|+1} \qquad (n \in \mathbb{N}),$$

$$\frac{n-1}{2} \leq K_n \leq \frac{1}{n} \sum_{k=0}^{n-1} M_{|k|+1} \qquad (n \in \mathbb{P}).$$

For Vilenkin system $n = D_n$ for every $n \in \mathbb{N}$ and $\frac{n-1}{2} = K_n$ for every $n \in \mathbb{P}$ hold, because of $n = D_n(0) \geq |D_n(x)|$ for all $x \in G_m$ and $n \in \mathbb{N}$.

Table 2 contains the values of a possible system for the *symmetric group $\mathcal{S}_3$* (for details see [10]).

A part of the sequence D_n from this system can be seen in Figure 1, and a part of the sequence K_n is shown in Figure 2 on the complete product of $\mathcal{S}_3$. This non-commutative system is a good example for nontrivial cases of our theorems.

	e	(12)	(13)	(23)	(123)	(132)
φ^0	1	1	1	1	1	1
φ^1	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
φ^3	1	-1	-1	-1	1	1
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$

Table 2. A possible system for $\mathcal{S}_3$

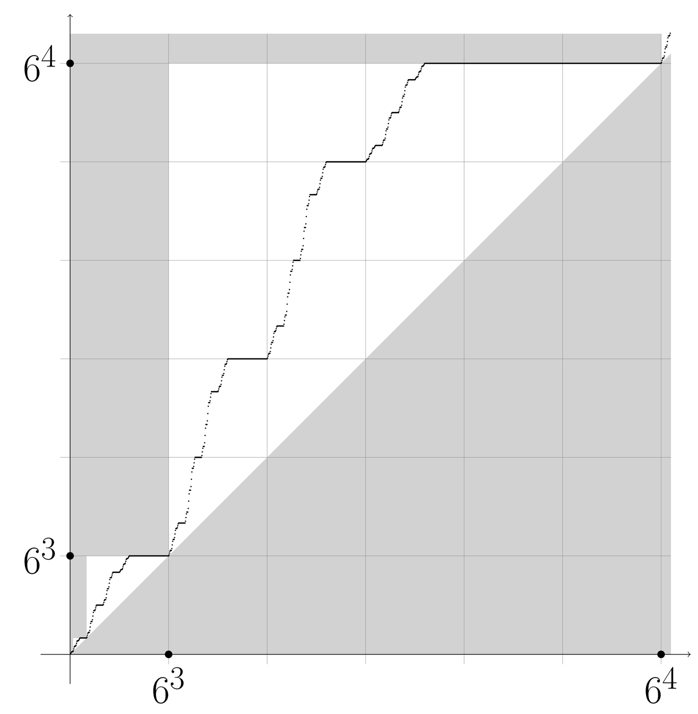

Figure 1. $n \leq D_n \leq 6^{|n|+1}$

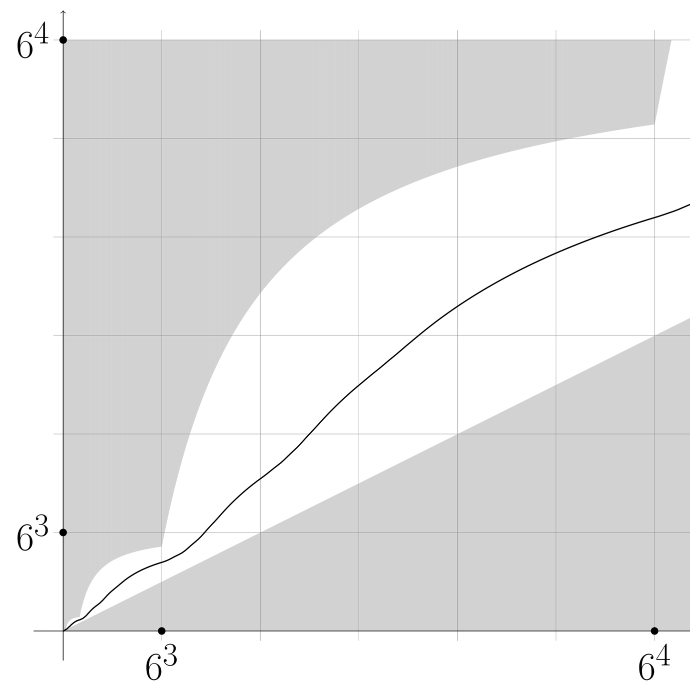

Figure 2. $\frac{n-1}{2} \leq K_n \leq \frac{1}{n} \sum_{k=0}^{n-1} 6^{|k|+1}$

Corollary 1 *[2] For every $n \in \mathbb{P}$*

$$1 \leq \frac{D_n}{n} \leq m_{|n|}, \qquad 1 \leq \frac{2}{n-1} K_n \leq \max_{1 \leq k < n} m_{|k|}.$$

Theorem 3 *[2] Let $n \in \mathbb{P}$. Equality $D_k = k$ holds for all $k \in \{0, \dots, n-1\}$ if and only if $K_n = \frac{n-1}{2}$.*

Theorem 4 *[2] The sequence D_n is increasing, while the sequence K_n is strictly increasing.*

Nevertheless, if $\inf_{x \in G_m} |\psi_n(x)| > 0$, then $D_n < D_{n+1}$.

But what is outside the diagonal?

Theorem 5 *[3] Let $x \in I_A(y) \backslash I_{A+1}(y)$, where $x, y \in G_m,\ A \in \mathbb{N}$. Then there is a constant C_m such that*

$$|D_n(y, x)| \leq C_m M_A.$$

Corollary 2 *[3] Let $x, y \in G_m$ and $x \neq y$. Then*

$$\sup_{n \in \mathbb{N}} |D_n(y, x)| < \infty.$$

On the other side, for any $x \in G_m$

$$\sup_{n \in \mathbb{N}} |D_n(x, x)| = \infty.$$

Bibliography

[1] Agaev, G. H., Vilenkin, N. Ja. and Dzhafarli, G. M. and Rubinstein, A. I., *Multiplicative systems of functions and harmonic analysis on 0-dimensional groups*, Elm Publisher, Baku, (1981)

[2] Blahota, I., *On the maximal value of Dirichlet and Fejér kernels with respect to the Vilenkin-like space*, Publicationes Mathematicae Debrecen, 80/3-4, 503–513 (2012)

[3] Blahota, I., *On the L_1 norm of the weighted maximal function of kernels with respect to the Vilenkin-like space*, Mathematica Pannonica, 23/1, 1–9 (2012)

[4] Gát, G. *On almost even arithmetical functions via orthonormal systems on Vilenkin groups*, Acta Arithmetica, 49 (2), 105–123 (1991)

[5] Gát, G., *On $(C, 1)$ summability of integrable functions on compact totally disconnected spaces*, Studia Mathematica, 144 (2), 101–120 (2001)

[6] Gát, G., *On the L^1 norm of the weighted maximal function of the Walsh-Kaczmarz-Dirichlet kernels*, Acta Academiae Paedagogicae Agriensis, Sectio Matematicae, 30, 55–66 (2003)

[7] Schipp, F., Wade, W. R., *Transforms on normed fields*, Janus Pannonius Tudományegyetem, Pécs (1995)

[8] Schipp, F., Wade, W. R., and Simon, P., *Walsh series. An Introduction to dyadic harmonic analysis*, Adam Hilger, Bristol and New York (1990)

[9] Taibleson, M. H., *Fourier Analysis on Local Fields*, Princeton University Press, Princeton (1975)

[10] Toledo, R., *On the maximal value of Dirichlet kernels with respect to representative product systems*, Rendiconti del Circolo Matematico di Palermo, Serie II., 82, 431–447 (2010)

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