Pseudoconnections on an Almost Complex Manifold

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The following facts are well-known:
1. On an almost complex manifold \((M, J)\) there exists a torsion-free almost complex linear connection iff the almost complex structure \(J\) is torsion-free.
2. On an almost Hermitian manifold \((M, J, g)\) the Levi–Civita connection is almost complex iff the manifold is Kählerian.

We shall study analogous problems for pseudoconnections. The class of almost complex pseudoconnections has been studied by Italian mathematicians and they have given a necessary and sufficient condition for Problem 1 in case of the pseudoconnections ([5, Prop. 4.6]) and they have published some partial results for Problem 2 ([4]).

In this paper we shall complete the answer to Problem 2 and give some other remarks on these topics.

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1. Basic notations

In this paper differentiable manifolds are of finite dimension, of Hausdorff-type, and of countable base. All manifolds, functions, and maps are \( C^\infty \). \( C^\infty (M) \) denotes the algebra of real functions on the manifold \( M \) and \( \tau_M \) means the tangent bundle of \( M \) with bundle projection \( \pi \) and \( \pi^* (\tau_M) \) denotes the pullback bundle of \( \tau_M \) with respect to (w.r.t.) \( \pi \). The module of the sections of \( \tau_M \) is denoted by \( \mathfrak{X}(M) \).

For the \( C^\infty (M) \)-modules \( M_0, M_1, \ldots, M_k \) \( \text{Hom}(M_1, \ldots, M_k; M_0) \) denotes the module of multilinear maps \( M_1 \times \ldots \times M_k \to M_0 \).

Concerning further terminology we refer to the monograph [8].

2. Preliminaries

A. Pseudoconnections ([2]) By a pseudoconnection on a manifold \( M \) we mean a pair \((\nabla, A)\), where \( A \in \text{End}(\tau_M) \) is a strong bundle endomorphism of the tangent bundle of \( M \), and \( \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \) is a map, which is \( C^\infty (M) \)-linear in its first variable, additive in its second variable, and the property

\[
\nabla_X fY = (AX)fY + f\nabla_X Y \quad (f \in C^\infty (M))
\]

is also satisfied.

The covariant derivatives of the tensor fields

\[
K \in \text{Hom} \left( \mathfrak{X} (M), \ldots, \mathfrak{X} (M); \mathfrak{X}(M) \right)
\]

and

\[
F \in \text{Hom} \left( \mathfrak{X} (M), \ldots, \mathfrak{X} (M); C^\infty (M) \right)
\]

are given by the expressions:

\[
(\nabla_X K)(Y_1, \ldots, Y_p) = \nabla_X K(Y_1, \ldots, Y_p) - \sum_{i=1}^p K(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_p)
\]
and

\[ (\nabla_X K)(Y_1, \ldots, Y_p) = (AX)F(Y_1, \ldots, Y_p) - \sum_{i=1}^{p} F(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_p) \]

respectively. On the base of these formulas a pseudoconnection \((\nabla, A)\) is almost complex on an almost complex manifold \((M, J)\) if \(\nabla_X J = 0\) (\(\forall X \in \mathfrak{X}(M)\)) and it is metrical on a Riemannian manifold \((M, g)\) if \(\nabla_X g = 0\) (\(\forall X \in \mathfrak{X}(M)\)).

**B. The twisted Lie-bracket and the operator** \(d_A\) ([9] [5]) If \(A \in \text{End}(\tau_M)\), then \(L_A(X, Y) = [AX, Y] + [X, AY] - A[X, Y]\) is the twisted Lie-bracket of vector fields \(X, Y\). Applying this bracket, we call the tensor \(\Sigma \in \text{Hom}(\mathfrak{X}(M), \mathfrak{X}(M); \mathfrak{X}(M))\), \(\Sigma(X, Y) = \nabla_X Y - \nabla_Y X - L_A(X, Y)\) the torsion tensor of the pseudoconnection \((\nabla, A)\).

We denote by \(\mathcal{D}^r(M)\) the space of all exterior differential forms of degree \(r\) on \(M\), and let \(\mathcal{D}(M) = \sum_{0}^{\dim M} \mathcal{D}^r(M)\). \(d_A\) denotes the antiderivation of degree \(1\) of \(\mathcal{D}(M)\) which acts on \(\mathcal{D}^0(M)\) and on \(\mathcal{D}^1(M)\) respectively, in the following way:

\[ (d_A f)(X) = (AX)f \quad (f \in \mathcal{D}^0(M) \equiv C^\infty(M)) \]

\[ (d_A \omega)(X, Y) = \frac{1}{2}(AX \omega(Y) - AY \omega(X) - \omega(L_A(X, Y))) \quad (\omega \in \mathcal{D}^1(M)). \]

**C. The Nijenhuis operator** Let \(K : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)\) be an arbitrary \(\mathbb{R}\)-linear map, and \(C \in \text{End}(\tau_M)\). We define the map \(K^C : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)\), as follows:

\[ K^C(X, Y) = K(CX, CY) + C^2K(X, Y) - CK(CX, Y) - CK(X, CY). \]

For \(K(X, Y) = [X, Y]\) \(2K^C\) is the Nijenhuis torsion \(N_C\) of \(C\) and the operator \(N^C : K \mapsto K^C\) is called **Nijenhuis operator**.

In the following we shall consider \(C = J\) (\(J\) is a fixed almost complex structure on \(M\).) Then \((L_A)^J \equiv L_A^J\) is a \(C^\infty(M)\)-bilinear map for all \(A \in \text{End}(\tau_M)\) (i.e. a tensor field) and the following simple relation holds:

\[ L_A^J(X, Y) = N_AJ(JX, Y) + N_AJ(X, JY) + \frac{1}{2} AN_J(X, Y) \]

(where \(N_AJ\) means the Nijenhuis torsion of the operator pair \((A, J)\) [10, vol. I ch.1 §3]).
3. The problem for non metrical pseudoconnections

M. Falcitelli and A. M. Pastore have proved in [6] that if the pseudoconnection \((\nabla, A)\) is almost complex on an almost complex manifold \((M, J)\) then
\[ \Sigma^J(X, Y) = -L_A^J(X, Y). \]
(This is an equivalent formalization of Falcitelli-Pastore's Proposition [6, Prop. 2.3], adapted to our above formalism on the base (1).) It follows: if there exists an almost complex pseudoconnection \((\nabla, A)\) on \((M, J)\) with vanishing torsion tensor field, then \(L_A^J = 0\). The basic result is that the reversed proposition is also true:

**Theorem 1.** (equivalent from of [5, Prop. 4.6]) On an almost complex manifold \((M, J)\) there exists a torsion-free almost complex pseudoconnection \((\nabla, A)\) iff \(L_A^J = 0\).

**Alternative proof.** (Outline.) The following two Lemmas are fundamental in our proof.

**Lemma 1.** If \(T \in \text{Hom}(\mathfrak{X}(M), \mathfrak{X}(M); \mathfrak{X}(M))\), \(T\) is antisymmetric and \(T^J = 0\), then there exists an \(S \in \text{Hom}(\mathfrak{X}(M), \mathfrak{X}(M); \mathfrak{X}(M))\) such that \(S\) is symmetric and
\[
T(X, JY) - JT(X, Y) = S(X, JY) - JS(X, Y).
\]

**Lemma 2.** If \(\nabla^L\) denotes an almost complex linear connection on an almost complex manifold \((M, J)\) with torsion tensor \(8T\) or \((X, Y) = N_J(X, Y)\), then
\[
L_A^J(X, Y) =
\]
\[
= \left\{ (\nabla^L X A)Y - \nabla^L Y A X \right\}^J -
\]
\[
- \frac{1}{8} \{ N_J(A X, Y) + N_J(X, A Y) - A N_J(X, Y) \}^J.
\]

Now, let
\[
T(X, Y) =
\]
\[
= - \frac{1}{16} \{ N_J(A X, Y) + N_J(X, A Y) - A N_J(X, Y) \} +
\]
\[
+ \frac{1}{2} \left[ (\nabla^L X A) Y - (\nabla^L Y A) X \right].
\]
This $T$ is antisymmetric and by reason of Lemma 2 $T^J(X,Y) = 0$, so in view of Lemma 1 there exists a symmetric $S$ such that (2) holds. Then we construct the following $(\nabla, A)$:

$$\nabla_X Y = \nabla^L_A X Y + T(X,Y) - S(X,Y) \quad (X,Y \in \mathfrak{X}(M)).$$

This $(\nabla, A)$ is almost complex and torsion-free. ■

Our problem is now to find an endomorphism $A$ (or to find all endomorphisms $A$) which satisfy the condition $L^J_A = 0$ for a fixed $J$. This problem seems to be difficult in general, so we examine a special case only.

Let $B$ the real algebra of real $2 \times 2$-type matrices and let $B^t$ denote its subalgebra of triangular matrices. In the following $\hat{B}$ means $B$ or $B^t$. A $\hat{B}$ structure $\hat{\sigma}$ (or $\sigma$) on a manifold $M$ is a morphism of real algebras $\hat{\sigma} : \hat{B} \to \text{End}(\tau_M)$ such that $\hat{\sigma}(1) = id$. If the $B^t$ structure $\sigma^t$ is the restriction of a $B$-structure $\sigma$ we say: $\sigma$ is the only $B$ structure which extends $\sigma^t$ (cf. [1, Th. 2.3]).

A linear connection $\nabla$ on $M$ is said to be compatible with the $\hat{B}$ structure $\hat{\sigma}$ if each operator $\hat{\sigma}(a) a \in \hat{B}$ is parallel w.r.t. $\nabla$ (i.e. $\nabla_X \hat{\sigma}(a) = 0 \ \forall X \in \mathfrak{X}(M)$).

A $\hat{B}$ structure $\hat{\sigma}$ is said to be integrable if each point $p \in M$ has a neighbourhood $(U, \varphi)$ such that $d\varphi \circ \hat{\sigma}(a) \circ d\varphi^{-1}$ does not depend on its variable $x \in U$.

The giving of a $B^t$ structure $\sigma^t$ (resp. a $B$ structure $\sigma$) on $M$ is equivalent to the giving of two operators $P, R \in \text{End}(\tau_M)$ (resp. $\hat{P}, \hat{Q} \in \text{End}(\tau_M)$) such that $P^2 = id$, $R^2 = 0$ and $RP = -PR = R$ (resp. $\hat{P}^2 = \hat{Q}^2 = id$ and $\hat{P}Q + Q\hat{P} = 0$). ([1, Th. 2.1]). If there is given a $B^t$ structure $\sigma^t$ which can be extended to a $B$ structure $\sigma$, then $P = \hat{P}$ and $J = PQ$ is an almost complex structure on $M$. The integrability of $\sigma^t$ implies the integrability of $J$.

**Proposition 1.** If there is given an integrable $B^t$ structure $\sigma^t$ which is extended to a $B$ structure $\sigma$, then for every $a \in B$ there exists an almost complex torsion-free pseudoconnection $(\nabla, \sigma(a))$ w.r.t. $J = PQ$.

**Proof.** In this case there exists a (unique) torsion-free and flat linear connection $\nabla^L$ on $M$, which is compatible with $\sigma$ ([1, Th. 2.5]), consequently this $\nabla^L$ is almost complex. Then in view of Lemma 2 and integrability of $J$ we have

$$L^J_A(X,Y) \stackrel{N_j = 0}{=} \{ (\nabla^L_X \sigma^t(a)) Y - (\nabla^L_Y \sigma^t(a)) X \}^J.$$
Finally $\nabla^L \sigma^t = 0$, therefore $L_A^J = 0$. \qed

**Remark.** With the same hypothesis as in the Proposition above. The pseudoconnection $(\nabla, \sigma(a))$, where $\nabla_X Y = \nabla^L \sigma(a) X Y$, is torsion-free and almost complex for every $a \in B$. Indeed, in the proof of Theorem 1 $T(X, Y) = S(X, Y) = 0$.

We give an example for the case $N_J \neq 0$, too.

**Proposition 2.** If there is given a $B^t$ structure $\sigma^t$ which can be extended to a $B$ structure $\sigma$ and the linear connection $\nabla^L$ with torsion $\frac{1}{8}N_J$ is compatible with $\sigma$, then $L_Q^J = 0$.

**Proof.** A simple but long calculation gives that

$$\{N_J(AX, Y) + N_J(X, AY) - AN_J(X, Y)\}^J =$$

$$= -2J[N_J((A \times J)X, Y) + N_J(X, (A \times J)Y) + (A \times J)N_J(X, Y)],$$

where $A \times J = AJ + JA$. Combining this fact with Lemma 2, we obtain the relation:

$$L_Q^J(X, Y) =$$

$$= \left\{ (\nabla^L_X Y) Q - (\nabla^L_Y Q) X \right\}^J J +$$

$$\frac{1}{4}J[(Q \times J)N_J(X, Y) + N_J((Q \times J)X, Y) + N_J(X(Q \times J)Y)] =$$

$$\nabla^L_Q = \frac{1}{4}J[(Q \times J)N_J(X, Y) + N_J((Q \times J)X, Y) + N_J(X, (Q \times J)Y)].$$

But $Q(PQ) + (PQ)Q = -Q(QP) + P = -P + P = 0$. \qed

4. The problem for metrical pseudoconnections

Let $G$ be a Hermitian metric on an almost complex manifold $(M, J)$. Then there exists uniquely a torsion-free metric pseudoconnection $(\nabla, A)$ for every $A$, the so called Levi–Civita pseudoconnection ([7, Prop. 4.1]). We put the question whether or not this pseudoconnection is almost complex. According to Theorem 1 $L_A^J = 0$ is necessary for this. Now we formulate the following
Theorem 2. A necessary and sufficient condition that a torsion-free metric pseudoconnection \((\nabla, A)\) with endomorphism \(A\) satisfying the condition \(L_A^J = 0\) should be almost complex is that \(d_A \Phi = 0\), where \(\Phi(X, Y) = g(X, JY)\) is the fundamental 2-form.

Proof. To prove this theorem we use the line of reasoning known from the proof of the analogous theorem for linear connections ([10, vol. II Prop. 4.2]). The difference is in the technique of computation: we use the “twisted calculus” reviewed in our preliminaries.

To prove the sufficiency we observe that
\[
2g(\nabla_X Y, Z) = \]
\[
= AXg(Y, Z) + AYg(X, Z) - AZg(X, Y) +
\]
\[
+ g(L_A(X, Y), Z) + g(L_A(Z, X), Y) + g(X, L_A(Z, Y))
\]
\(([7, (4.1)])\). Taking into account this fact and the definition of \(d_A\) one can easily derive the following relation:
\[
2g(\nabla_X J)(Y, Z) =
\]
\[
= 3d_A \Phi(X, JY, JZ) - 3d_A \Phi(X, Y, Z) + g(L_A^J(Y, Z), JX) =
\]
\[
= 3d_A \Phi(X, JY, JZ) - 3d_A \Phi(X, Y, Z).
\]

Accordingly if \(d_A \Phi = 0\), then \(\nabla_X J = 0\).

The necessity will be based on the following

Lemma 3. \(3d_A \Phi(X, Y, Z) = \sigma\left\{ (\nabla_X \Phi)(Y, Z) \right\} \) (where \(\sigma\) means the cyclic sum w.r.t. \(X, Y, Z\)).

It is easy to see, that for an almost complex pseudoconnection
\[
(\nabla_X \Phi)(Y, Z) = - (\nabla_X g)(JY, Z)
\]
holds, i.e. for a metric almost complex pseudoconnection \(\nabla_X \Phi = 0\), consequently \(d_A \Phi = 0\). 

After these we give the following

Definition. We shall say that the quadruple \((M, g, J, A)\) is a twisted Kählerian manifold if \(J\) is an almost complex structure on \(M\), \(g\) is a Hermitian metric, \(A \in \text{End}(\tau_M)\), and \(L_A^J = 0\), \(d_A \Phi = 0\) hold.

Now we can state our Theorem 2 in a more attractive form: On a twisted Kählerian manifold the Levi–Civita pseudoconnection is almost complex.
Finally we give some examples for twisted Kählerian manifolds.

Let \((M, g, J)\) be a manifold with a fixed almost complex structure \(J\) and with a Hermitian metric \(g\). Let \(\mathcal{H} : \pi^* (\tau_M) \to \tau_M\) be a horizontal map. This \(\mathcal{H}\) determines the horizontal lift \(X^h\) of the vector field \(X \in \mathfrak{X}(M)\), and the operator \(h\) of the horizontal projection. Let us denote by \(X^v\) the vertical lift of \(X \in \mathfrak{X}(M)\) and by \(v\) the operator of the vertical projection w.r.t. \(\mathcal{H}\). For a fixed \(\mathcal{H}\) \(TM\) carries a Riemannian metric \(G\) defined by \(G(X^v, Y^v) = G(X^h, Y^h) = g(X, Y) \circ \pi, G(H^hY^v) = 0\). ([3] If \(\mathcal{H}\) is the horizontal map of the Levi–Civita connection on \((M, g)\), then this \(G\) is the Sasakian metric.)

Under the hypothesis above, the relations

\[ F(X^h) = X^v, \quad F(X^v) = -X^h; \quad \tilde{J}^h(X^h) = (JX)^v, \quad \tilde{J}^h(X^v) = (JX)^h \]

define two almost complex structures on \(TM\). For these structures we have the following

**Proposition 3.** If the Nijenhuis torsion \(N^\mathcal{H}\) of \(\mathcal{H}^1\) vanishes for horizontal lifts, i.e. if \(v[X^h, Y^h] = 0 \ \forall X, Y \in \mathfrak{X}(M)\), then \((TM, G, F, v)\) and \((TM, G, \tilde{J}^h, v)\) are twisted Kählerian manifolds.

**Proof.** Let us denote by \(J\) the almost complex structure \(F\) or \(\tilde{J}^h\). \(L^J_v\) is a tensor, so it is enough to prove its vanishing for vertical and horizontal lifts. It is easy to see that \(L_v(X^v, Y^v) = 0\) \((\sigma, \eta = h \text{ or } v)\):

\[
L_v(X^h, Y^h) = -v[X^h, Y^h] \overset{\mathcal{N}_X = 0}{=} 0, \\
L_v(X^h, Y^v) = h[X^h, Y^v] = 0 \quad ([X^h, Y^v] \text{ is vertical}), \\
L_v(X^v, Y^v) = v[X^v, Y^v] = 0 \quad ([X^v, Y^v] = 0).
\]

\(J\) maps a (vertical or horizontal) lift of a vector field on a (vertical or horizontal) lift, so \(L^J_v = 0\).

A similar method can be used to prove \(d_v \Phi = 0\):

\[
d_v \Phi(X^h, Y^h, Z^h) = 0, \\
d_v \Phi(X^v, Y^h, Z^h) = X^vG(Y^h, FZ^h) = 0 \quad (FZ^h \text{ is a vertical lift}), \\
d_v \Phi(X^h, Y^v, Z^v) = -Y^vG(X^h, FZ^v) + Z^vG(X^h, FY^v) = 0
\]

1. \(\frac{1}{2} N^\mathcal{H}(X, Y) = [hX, hY] + h[X, Y] - h[hX, Y] - h[X, hY]\)
(FZ', FY') are horizontal lifts and X'(f \circ \pi) = 0 \forall X \in \mathfrak{X}(M), f \in C^\infty(M),
d_v \Phi(X', Y', Z') = X'G(Y', FZ') - Y'G(X', FZ') + Z'G(X', FY') = 0
(FY' is a horizontal lift).

We remark that N_\kappa (X', Y') = 0 (\forall X, Y \in \mathfrak{X}(M)) is not an integrability condition of J, cf. [11].

References


