DIFFERENCE EQUATIONS AS A MODELLING TOOL IN SCHOOL MATHEMATICS

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Abstract. The use of difference and differential equations in the modelling is a topic usually studied by advanced students in mathematics. However difference and differential equations appear in the school curriculum in many direct or hidden way. Difference equations first enter in the curriculum when studying arithmetic sequences. Moreover Newtonian mechanics provides many examples for differential equations and numeric solution leads to difference equations which can be treated easily with Computer Algebra Systems or even by Dynamic Geometry Softwares. My hypothesis is that numerical methods supported by technology serves a tool which helps the early introduction of modelling concepts.

1. The key concept

Newton’s second law describes the relationship between the forces acting on a particle to the motion of the particle. The force might depend on time, position, velocity. In such cases, Newton’s law becomes a system of differential equations. Our problem is how to treat this system in the school with school mathematics at the age of 16-18.

One possible solution might be to transform the system to difference equations, i.e. we are looking for numerical solutions. R. Feynman used this concept in his famous lectures at Caltech to undergraduate students [4, Chapter 9. Newton’s laws of dynamics]:

\[ x(t + \varepsilon) = x(t) + \varepsilon v_x(t) \]
\[ v_x(t + \varepsilon) = v_x(t) + \varepsilon a_x(t) \]

Therefore, if we know both the \( x \) and \( v \) at a given time, we know the acceleration, which tells us the new velocity, and we know the new position—this is how the machinery works. The velocity changes a little bit because of the force, and the position changes a little bit because of the velocity.

L. Berg analysed didactic aspects of this approach and interpreted it as a change from continuous model to discrete model [1]. Computer algebra systems, dynamic geometry applications and even spreadsheet applications give a fresh look to this concept which can be easily introduced in schools.

In this paper I will focus on dynamic geometry system GeoGebra [5] and I will give some examples with computer algebra system Maxima, but almost everything in this paper can be transformed without difficulty to spreadsheet applications.

ZDM Subject Classification. I74, M54.

Key words and phrases. difference equations, Newtonian mechanics, Euler method.
2. Numerical solution of an initial value problem

Suppose we wish to numerically solve the initial value problem

\( \dot{y} = f(t, y), \quad y(t_0) = y_0 \)

on an interval \([t_0, b]\). By a numerical solution, we mean an approximation of the solution at a finite number of points, i.e.

\[ (t_0, y_0), (t_0 + \Delta t, y_1), (t_0 + 2\Delta t, y_2), \ldots (t_0 + n\Delta t, y_n), \]

where

\[ \Delta t = \frac{b - t_0}{n}. \]

To approximate the solution of (1) compute \(y_1, y_2, \ldots, y_n\) replacing \(y'(t_k)\) by its forward difference approximation

\[ y'(t_k) \approx \frac{y_{k+1} - y_k}{\Delta t} \]

i.e. using the difference equation

\[ y_{k+1} = y_k + \Delta t f. \]

The crucial point is the argument of \(f\) at the right-hand side of (2). The well-known Euler forward method uses \(f(t_k, y_k)\):

\[ y_{k+1} = y_k + \Delta t f(t_k, y_k), \quad t_k = t_0 + k\Delta t, \]

while the backward method takes \(f(t_{k+1}, y_{k+1})\). An obvious shortcoming of these methods is that they make the approximation based on information at the beginning/ending of the interval only. Consider \(f\) at both the beginning and ending of the time step and take the average of the two. Doing this produces a symmetric method represented by the equation:

\[ y_{k+1} = y_k + \Delta t \frac{1}{2} \left( f(t_{k+1}, y_{k+1}) + f(t_k, y_k) \right). \]

This is an implicit method which means that a system of equations must be solved for \(y_{k+1}\) at each step. To get around this difficulty (if necessary) we may use (3) to approximate \(y_{k+1}\) at the right-hand side of (4):

\[ y_{k+1} = y_k + \Delta t \frac{1}{2} \left( f(t_{k+1}, y_k + \Delta t f(t_k, y_k)) + f(t_k, y_k) \right). \]

(5) is known as improved Euler method.

Now, apply the above method to the equations of motion and let us suppose that the force depends only on the directional vector of the particle. \(m = 1\) in this paper.

\[ \begin{pmatrix} \mathbf{r}(t) \\ \dot{\mathbf{v}}(t) \end{pmatrix} = \begin{pmatrix} \dot{\mathbf{v}}(t) \\ \mathbf{F} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{r}(t_0) \\ \mathbf{v}(t_0) \end{pmatrix} = \begin{pmatrix} \mathbf{r}_0 \\ \mathbf{v}_0 \end{pmatrix}. \]

From the Euler forward method we find that

\[ \mathbf{r}_{k+1} = \mathbf{r}_k + \Delta t \mathbf{v}_k \]

(7)

\[ \mathbf{v}_{k+1} = \mathbf{v}_k + \Delta t \mathbf{F}_k. \]

(8)

It is possible to eliminate the velocity from the first equation. First, transform the index in (7)

\[ \mathbf{r}_{k+2} = \mathbf{r}_{k+1} + \Delta t \mathbf{v}_{k+1}. \]
Subtraction (9) - (7) leads to

\[ r_{k+2} - r_{k+1} = r_{k+1} - r_k + \Delta t (v_{k+1} - v_k). \]  

Substituting \( v_{k+1} - v_k \) from (8) gives

\[ r_{k+2} = 2r_{k+1} - r_k + \Delta t^2 F_k. \]

\( r_0 \) is given and we get \( r_1 \) from (7)

\[ r_1 = r_0 + \Delta tv_0. \]

It follows from the improved Euler method that

\[ r_{k+1} = r_k + \frac{1}{2} \Delta t (v_k + v_{k+1}) \]  

\[ v_{k+1} = v_k + \frac{1}{2} \Delta t (F_k + F_{k+1}). \]

To avoid implicit equations we approximate \( v_{k+1} \) in the first equation

\[ r_{k+1} = r_k + \Delta t \frac{1}{2} v_k + \Delta t \frac{1}{2} (v_k + \Delta t F_k) \]  

\[ = r_k + \Delta t v_k + \frac{1}{2} \Delta t^2 F_k. \]

After applying the ‘index transformation trick’ this simplifies to

\[ r_{k+2} = 2r_{k+1} - r_k + \Delta t^2 F_{k+1}. \]

We get \( r_1 \) from (13)

\[ r_1 = r_0 + \Delta tv_0 + \frac{1}{2} \Delta t^2 F_0. \]

Now apply the above methods to the equation of the harmonic linear oscillator. The corresponding differential equations are

\[ x' = v, \quad v' = -Dx, \quad x(0) = x_0, \quad v(0) = v_0 \]

where \( D \) is a positive constant. In this case we know the exact solution, namely

\[ x(t) = A \sin \left( \sqrt{Dt} + \phi \right), \]

\[ A = \sqrt{\frac{x_0^2}{v_0^2} + \frac{v_0^2}{D}}, \quad \tan \phi = \frac{\sqrt{D}x_0}{v_0} \text{ for } v_0 \neq 0, \quad \phi = \frac{\pi}{2} \text{ for } v_0 = 0. \]

Figure 1 demonstrates the result of the Euler and improved Euler methods. This figure implies a very natural question: how to choose the appropriate model. The general principle is that we must check our model somehow. In the case of the harmonic oscillator there is a possibility for checking which is available also in the classroom. We compare the maximum displacement from the numeric calculation with the theoretic value of the amplitude, obtained from the energy conservation law. Obviously, Euler method fails this test in this case because the error is not acceptable for this step-size. It is an important problem whether the Euler method converges to the exact solution or not when \( \Delta t \to 0 \). The general answer is negative, see [6].

Note that the specialisation of the implicit system (12) gives

\[ x_{k+1} = x_k + \frac{1}{2} \Delta t v_{k+1} \]

\[ x_{k+1} = x_k + \frac{1}{2} \Delta t v_k \]  

\[ \frac{1}{2} D \Delta tx_{k+1} + v_{k+1} = -\frac{1}{2} D \Delta tx_k + v_k. \]
which is a system of linear equations for $x_{k+1}$ and $v_{k+1}$. The solution is

$$x_{k+2} = \frac{1 - \rho}{1 + \rho} x_{k+1} - x_k,$$

$$x_1 = \frac{1 - \rho}{1 + \rho} x_0 + \frac{\Delta t}{1 + \rho} v_0,$$

where $\rho = \frac{1}{4} D \Delta t^2$. The advantage of this symmetric model is that it satisfies the energy conservation law ([2, 1]), see Figure 2.
For didactic reason, we would like to avoid mathematical difficulties and everything should be motivated physically.

Our aim is to find the path of the particle under the effect of the force depending only on position. We know the force law, the initial position and the initial velocity. We divide the time interval into small intervals of the same length $\Delta t$ and suppose that the force is constant in the interval $[t_k, t_{k+1} = t_k + \Delta t]$. (Students know this type of motion, e.g. free fall is a good case.) The problem is to get $(P_{k+1}, v_{k+1})$ from $(P_k, v_k)$. In the first step we are looking for the new position $P_{k+1}$. At this point we know the force only in $P_k$. The shift of the position is known from the ‘constant force’ situation, i.e. put $F_k = F_{k+1}$ in (12):

$$(CP1) \quad P_{k+1} = P_k + \Delta t v_k + \frac{1}{2} \Delta t^2 F_k.$$  

Note, this equation is the same as (13). In the second step we are looking for $v_{k+1}$. At this stage we know $F(P_k)$ and even $F(P_{k+1})$ thus it is reasonable to get the average of $F(P_k)$ and $F(P_{k+1})$ on the interval $[t_k, t_{k+1}]$, which gives the relation

$$(CP2) \quad v_{k+1} = v_k + \Delta t \left( \frac{1}{2} F(P_k) + F(P_{k+1}) \right).$$

From relations (CP1), (CP2) we get a very simple construction pattern:

$$(P_k, v_k, F_k) \xrightarrow{(CP1)} P_{k+1} \xrightarrow{(CP2)} v_{k+1},$$

see Figure 3.

Construction patterns (CP1), (CP2) fit very well to dynamic geometry systems. For spreadsheet applications it is more convenient to eliminate the velocity with the ‘index transformation trick’, getting (14).

4. THE TERMINAL SPEED OF THE PARACHUTIST

The terminal velocity of a falling body occurs during free fall when a falling body experiences zero acceleration. This is because of the retarding force known as air resistance. This upward force will eventually balance the falling body’s weight.

Our case-study is the following. A parachutist of mass $m$ falls freely until his parachute opens. When it is open she/he experiences an upward resistance $kv$ where $v$ is her/his speed and $k$ is a positive constants. Draw the velocity-time diagram and determine the terminal velocity.

The equation of the motion is

$$(18) \quad m v' = mg - kv, \quad v(0) = v_0.$$
The standard reasoning for the terminal velocity question is the following. When the parachutist reaches the terminal velocity her/his acceleration is zero, thus

\[ v' = 0 \implies v = \frac{mg}{k}. \]

**Remark.** The exact solution of the initial value problem (18) is

\[ v(t) = a \cdot \exp \left( \frac{-kt}{m} \right) + \frac{mg}{k}, \]

where constant \( a \) is determined by the initial velocity \( v_0 \). The limit

\[ \lim_{t \to \infty} v(t) = \frac{mg}{k} \]

yields to the terminal velocity.

In the classroom we change to discrete model, i.e. transform (18) to difference equation. The simplest model is the Euler forward model:

\[ \frac{v_{k+1} - v_k}{\Delta t} = -\frac{k}{m} v_k + g, \]

thus

\[ v_{k+1} = \left( 1 - \frac{k}{m} \Delta t \right) v_k + g \Delta t. \]

The equation for the symmetric model is given by

\[ \frac{v_{k+1} - v_k}{\Delta t} = -\frac{k}{m} \left( \frac{v_{k+1} + v_k}{2} \right) + g. \]

It is then a simple matter to calculate

\[ v_{k+1} = v_k \frac{1 - \rho}{1 + \rho} + \frac{g \Delta t}{1 + \rho}, \quad \rho = \frac{k \Delta t}{2m}. \]

The theoretic value of the terminal velocity obtained directly from (18) tests both models to be acceptable (Figure 4).
5. A GeoGebra application: the Kepler problem

Kepler’s laws are concerned with the motion of the planets around the Sun. The Sun (S) is supposed to be in a constant position and the planet (P) moves under the effect of central force:

$$\mathbf{F}(P) = \frac{-\kappa}{d(S, P)^3}(S - P).$$

First we apply directly the construction principle. GeoGebra’s algebraic input is a convenient method to feed (CP1), (CP2) directly in order to get $P_1$ and $v_1$ from $(P_0, v_0)$. (These are the sixth and seventh lines of the construction protocol, see Figure 5). After the seventh step we created a new tool called EulerMethod (Figure 6). In my worksheet $\Delta t$ is given by a slider. The result is in the Figure 7 (left part). What is more, GeoGebra offers a ‘bonus’. We may demonstrate Kepler’s first law if we construct the ellipse determined by the first five points (right part of the figure.) Demonstration of Kepler’s second law is very easy, too.

The approach described in the previous paragraph has (at least) one disadvantage: it is uncomfortable to create numerous points, which could be necessary for small $\Delta t$. In the second approach we deal with this shortage. The ‘index transformation trick’ leads to (11) which simplifies the construction pattern, i.e. we can get $P_{k+2}$ directly from $(P_k, P_{k+1})$. (However, we lost the direct physical motivation.) The new spreadsheet feature of GeoGebra supports this reasoning very well.
After creation of $P_0$ and $P_1$ we apply (11) only one times to get $P_2$ then with the automatic filling in functionality we may derive as many new points as we like, see Figure 8.

6. Conclusion

Early introduction of topics related to difference equations should gain more interest in the school mathematics [3]. This idea is not a new discovery, see e.g.
L. Berg’s papers in the seventies. Nowadays, technology gives a new tool to this concept, not only for the easy numerical calculations but for the graphical representations, too. In my paper I demonstrated how dynamic geometry applications may be used to solve equations of motion numerically. The advantage of the concept is that we can choose complex problems and we can treat them without black boxes. An obvious disadvantage is that the discrete model for the problem is not unique (even it depends on the step-size), so we must test our results.

References


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